# Melville Senior High School

**Semester Two Examination, 2020**

**Question/Answer booklet**

# MATHEMATICS SPECIALIST UNITS 3&4

## Section One: Calculator-free

**SOLUTIONS**

WA student number: In figures | | | | | | | | |

In words ______________________________

Your name ______________________________

## Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Number of additional answer booklets used (if applicable): ☐

## Materials required/recommended for this section

***To be provided by the supervisor***
This Question/Answer booklet
Formula sheet

***To be provided by the candidate***
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

## Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

## Structure of this paper

| Section | Number of questions available | Number of questions to be answered | Working time (minutes) | Marks available | Percentage of examination |
|---|---|---|---|---|---|
| Section One: Calculator-free | 8 | 8 | 50 | 52 | 35 |
| Section Two: Calculator-assumed | 13 | 13 | 100 | 98 | 65 |
| | | | | **Total** | 100 |

## Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

**Section One: Calculator-free** **35% (52 Marks)**

This section has **eight** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

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**Question 1** **(6 marks)**

The function $f$ is defined by

$$f(x) = \frac{6}{2x + x^3} = \frac{ax}{x^2 + 2} + \frac{b}{x}.$$

(a) Determine the value of the constant $a$ and the value of the constant $b$. (3 marks)

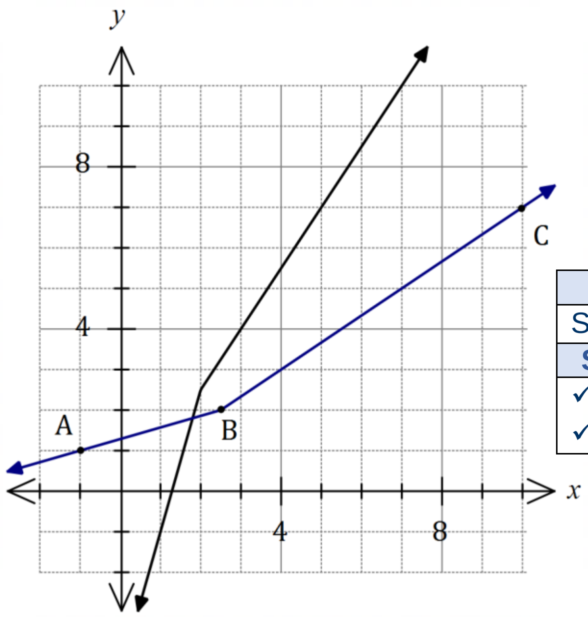
| Solution |
|---|
| $\frac{ax}{x^2+2} + \frac{b}{x} = \frac{ax^2 + bx^2 + 2b}{x(x^2+2)} = \frac{6}{2x + x^3}$ <br> $2b = 6 \Rightarrow b = 3$ <br> $a + b = 0 \Rightarrow a = -3$ |
| **Specific behaviours** |
| ✓ combines fractions correctly <br> ✓ correct value of $a$ <br> ✓ correct value of $b$ |

(b) Hence, or otherwise, determine the value of $\int_1^2 f(x)\,dx$ in simplest form. (3 marks)

| Solution |
|---|
| $\int_1^2 \frac{3}{x} - \frac{3x}{x^2+2}\,dx = \left[3\ln x - \frac{3}{2}\ln(x^2+2)\right]_1^2$ <br> $= 3\ln 2 - \frac{3}{2}\ln 6 - \left(3\ln 1 - \frac{3}{2}\ln 3\right)$ <br> $= 3\ln 2 - \frac{3}{2}\ln 2 - \frac{3}{2}\ln 3 - 0 + \frac{3}{2}\ln 3$ <br> $= \frac{3}{2}\ln 2$ |
| **Specific behaviours** |
| ✓ expresses integral using partial fractions <br> ✓ antidifferentiates <br> ✓ substitutes and simplifies |

**Question 2** **(6 marks)**

The function $f$ is defined by $f(x) = 2.5(x-1) - |x-2|$. The graph of $y = f(x)$ is shown below.




| Solution (b) |
|---|
| See graph |
| **Specific behaviours** |
| ✓ segment $AB$ |
| ✓ segment $BC$ |

(a) State the value of $f^{-1}(7)$. (1 mark)

| Solution |
|---|
| $f^{-1}(7) = 5$ |
| **Specific behaviours** |
| ✓ correct value |

(b) Sketch the graph of $y = f^{-1}(x)$ on the axes above. (2 marks)

(c) Solve $f(x) = f^{-1}(x)$. (3 marks)

| Solution |
|---|
| Intersect on the line $y = x$ when $x < 2$ and so <br> $$f(x) = 2.5(x-1) + (x-2)$$ $$= 3.5x - 4.5$$ <br> $$3.5x - 4.5 = x$$ $$x = \frac{9}{5} = 1.8$$ |
| **Specific behaviours** |
| ✓ simplifies required part of $f$ <br> ✓ indicates solution will lie on $y = x$ <br> ✓ correct solution |

**Question 3** **(8 marks)**

The function $f$ is defined by $f(x) = x^2 - 4x + 7,\ x \geq 0$.

(a) Determine the range of $f$. (3 marks)

| Solution |
|---|
| $f = (x-2)^2 + 3$<br><br>$f$ has minimum at $(2, 3)$<br><br>$R_f = \{y: y \geq 3\}$ |
| **Specific behaviours** |
| ✓ differentiates or completes square<br>✓ shows minimum within defined domain of $f$<br>✓ correct range |

The function $g$ is defined by $g(x) = 8 - \sqrt{x+6},\ x \geq -6$.

(b) Determine an expression for $g \circ f(x)$. (2 marks)

| Solution |
|---|
| $g \circ f(x) = 8 - \sqrt{(x^2-2)+3+6}$ or $8 - \sqrt{x^2 - 4x + 7 + 6}$<br>$= 8 - \sqrt{(x^2-2)+9}$ or $8 - \sqrt{x^2 - 4x + 13}$ |
| **Specific behaviours** |
| ✓ indicates correct method<br>✓ correct composite function, simplified |

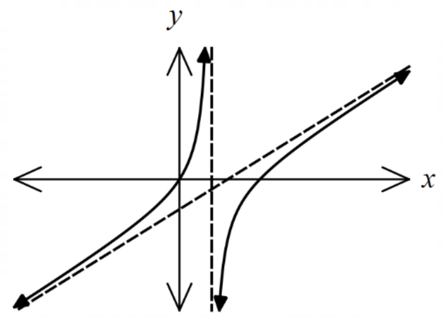
(c) Determine the domain and range of $g \circ f(x)$. (3 marks)

| Solution |
|---|
| Radicand of $g \circ f(x)$ has minimum when $x = 2$.<br><br>$g \circ f(2) = 5 \Rightarrow R_{gf} = \{y: y \leq 5\}$.<br><br>$D_{gf} = D_f = \{x: x \geq 0\}$ |
| **Specific behaviours** |
| ✓ indicates maximum of $g \circ f(x)$ when radicand minimum<br>✓ correct range of composite function<br>✓ correct domain |

**Question 4** **(7 marks)**

Let $f(x) = \dfrac{x(2x-5)}{x-1}$.

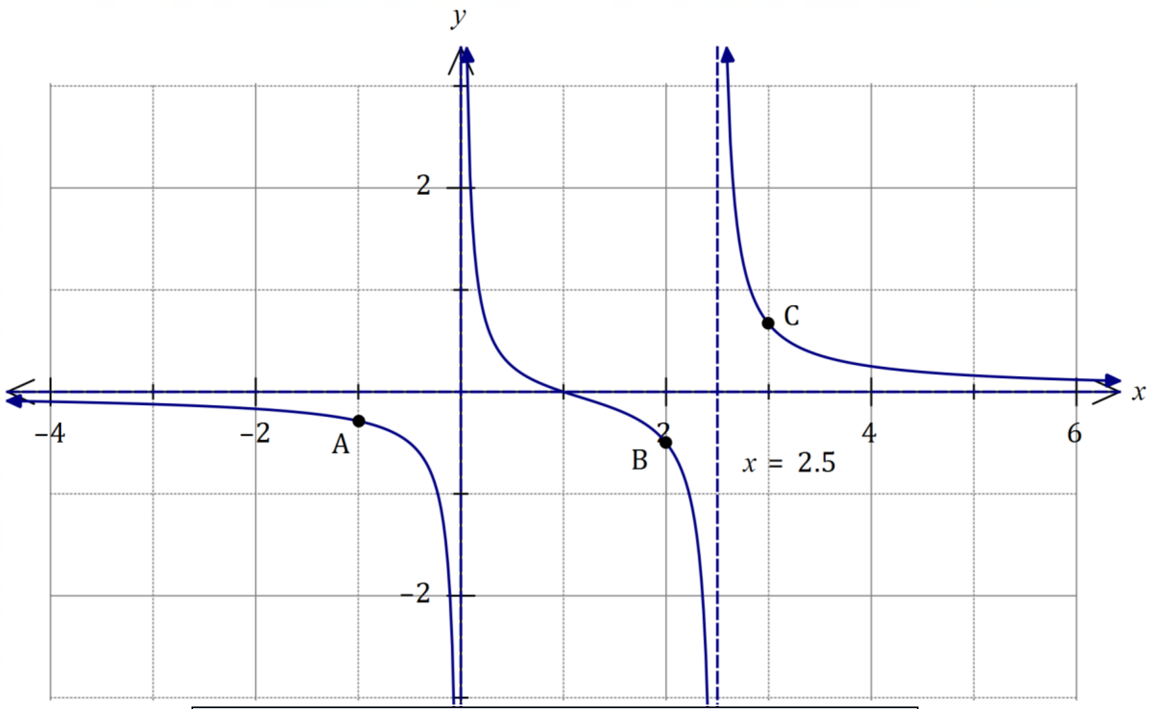
The graph of $y = f(x)$ is shown at right.




(a) Determine the equation of each asymptote shown on the graph of $y = f(x)$. (3 marks)

| Solution |
|---|
| $f(x) = \dfrac{2x^2 - 2x - 3x + 3 - 3}{x-1} = \dfrac{2x(x-1)}{x-1} - \dfrac{3(x-1)}{x-1} - \dfrac{3}{x-1}$ <br> $= 2x - 3 - \dfrac{3}{x-1}$ <br><br> Asymptotes: $y = 2x - 3$ and $x = 1$. |
| **Specific behaviours** |
| ✓ attempts to write $f(x)$ to expose slanted asymptote <br> ✓ correct slant asymptote <br> ✓ vertical asymptote |

(b) On the axes below, sketch the graph of $y = \dfrac{x-1}{x(2x-5)}$. (4 marks)




| Solution |
|---|
| See graph |
| **Specific behaviours** |
| ✓ asymptotic behaviour to L of $x = 0$ <br> ✓ asymptotic behaviour to R of $x = 2.5$ <br> ✓ asymptotic behaviour between $x = 0$ and $x = 2.5$ <br> ✓ through $(1, 0)$ and close to $A$, $B$ and $C$ |

**Question 5** **(5 marks)**

Use the substitution $x = \sqrt{3}\tan\theta$ to evaluate $\int_0^1 \frac{2}{x^2+3}\,dx$

| Solution |
| --- |
| $dx = \sqrt{3}\sec^2\theta\,d\theta$<br><br>$x = 0, \theta = 0;\ \ x = 1, \theta = \frac{\pi}{6}$<br><br>$x^2 + 3 = 3\tan^2\theta + 3$<br>$= 3(\tan^2\theta + 1)$<br>$= 3\sec^2\theta$<br><br>$\int_0^1 \frac{2}{x^2+3}\,dx = \int_0^{\frac{\pi}{6}} \frac{2\sqrt{3}\sec^2\theta}{3\sec^2\theta}\,d\theta$<br>$= \int_0^{\frac{\pi}{6}} \frac{2\sqrt{3}}{3}\,d\theta$<br>$= \left[\frac{2\sqrt{3}\theta}{3}\right]_0^{\frac{\pi}{6}}$<br>$= \frac{\sqrt{3}\pi}{9}$ |
| **Specific behaviours** |
| ✓ obtains $dx$ in terms of $d\theta$<br>✓ changes limits<br>✓ simplifies integrand<br>✓ antidifferentiates integrand<br>✓ evaluates definite integral |

**Question 6** **(5 marks)**

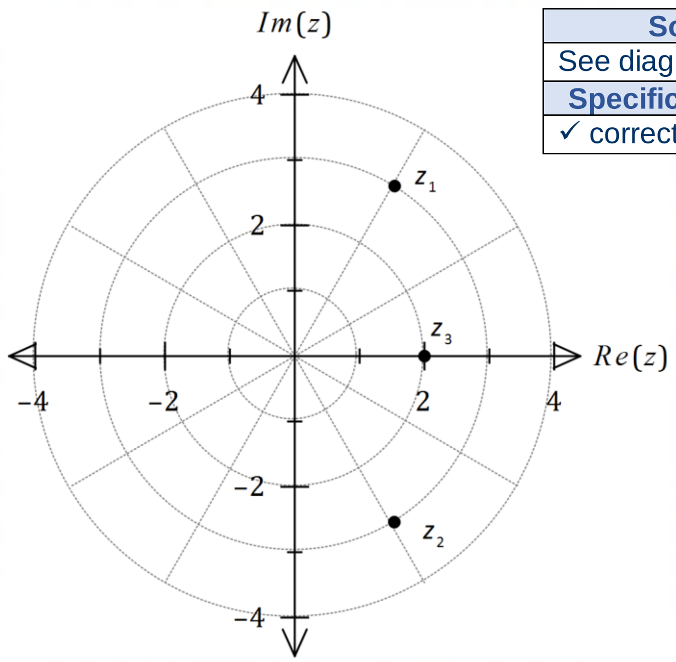
Consider the equation $z^3 - 5z^2 + 15z - 18 = 0,\ z \in \mathbb{C}$.

One root of the equation is $z = 3\,\text{cis}\left(\frac{\pi}{3}\right)$.

(a) Solve the equation, giving all solutions in Cartesian form. (4 marks)

| Solution |
|---|
| Equation has real coefficients - complex roots will occur as conjugate pairs:<br>$$z_1 = 3\,\text{cis}\left(\frac{\pi}{3}\right) = \frac{3}{2}(1+\sqrt{3}i), \qquad z_2 = 3\,\text{cis}\left(-\frac{\pi}{3}\right) = \frac{3}{2}(1-\sqrt{3}i)$$<br>Since the equation must equal $(z-z_1)(z-z_2)(z-z_3)$ then<br>$$(-z_1)(-z_2)(-z_3) = -18$$<br>But $(-z_1)(-z_2) = z_1 z_2 = 3\,\text{cis}\left(\frac{\pi}{3}\right) \times 3\,\text{cis}\left(-\frac{\pi}{3}\right) = 9$ and so $9(-z_3) = -18 \Rightarrow z_3 = 2$<br>Hence solutions are $z = 2,\ \frac{3}{2}(1+\sqrt{3}i),\ \frac{3}{2}(1-\sqrt{3}i)$. |
| **Specific behaviours** |
| ✓ second root in polar form<br>✓ expresses both polar roots in Cartesian form<br>✓ uses product of roots and $z^0$ term<br>✓ third solution |

(b) Locate all the roots of the equation on the Argand diagram below. (1 mark)




| Solution |
|---|
| See diagram |
| **Specific behaviours** |
| ✓ correct locations |

**Question 7** **(6 marks)**

The equations of three planes are $x + ay + bz = 1$, $2x - 3y + z = 5$ and $2x - y + 3z = -1$, where $a$ and $b$ are integer constants.

Elimination can be used to reduce the system of equations to

$$\begin{aligned} x + ay + bz &= 1 \\ (2a+3)y + (2b-1)z &= -3 \\ (b-a-2)z &= 3a+3 \end{aligned}$$

(a) Determine any necessary restrictions on the value of $a$ and/or the value of $b$ for the system of equations to have

(i) a unique solution. (1 mark)

| Solution |
|---|
| Require $b - a - 2 \neq 0 \Rightarrow b \neq a + 2$. |
| **Specific behaviours** |
| ✓ correct restriction |

(ii) no solutions. (1 mark)

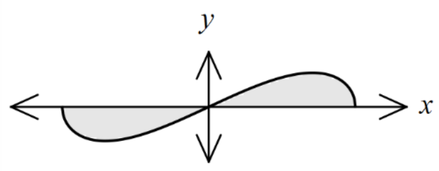
| Solution |
|---|
| Require $b - a - 2 = 0$ and $3a + 3 \neq 0$.<br><br>Hence $a \neq -1$ and $b = 2 + a$. |
| **Specific behaviours** |
| ✓ correct restrictions |

(b) For a particular value of $a$ and value of $b$, the three planes intersect in a straight line. Determine the vector equation of this line. (4 marks)

| Solution |
|---|
| Require $a = -1$ , $b = 1$ and let $z = \lambda$.<br><br>Using reduced equation (2) $y + z = -3$:<br><br>$y = -z - 3 = -\lambda - 3$<br><br>Using reduced equation (1) $x - y + z = 1$:<br><br>$x = y - z + 1 = -\lambda - 3 - \lambda + 1 = -2\lambda - 2$<br><br>Hence $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2\lambda - 2 \\ -\lambda - 3 \\ \lambda \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$ |
| **Specific behaviours** |
| ✓ indicates required value of $a$ and value of $b$<br>✓ expresses $z$ and $y$ in terms of parameter<br>✓ expresses $x$ in terms of parameter<br>✓ correct vector equation (many alternatives exist...) |

**Question 8** **(9 marks)**

(a) The graph of the curve $y = 6x\sqrt{4 - x^2}$ is shown at right.

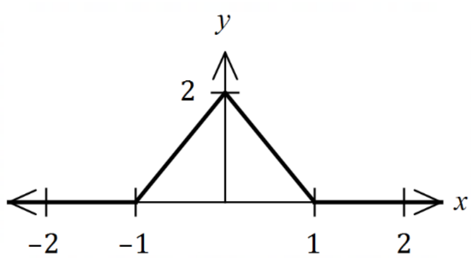



Using the substitution $u = 4 - x^2$, or otherwise, determine the area between the curve and the $x$-axis.

(6 marks)

| Solution |
|---|
| $y = 0 \Rightarrow x = 0, \pm 2$ <br> $x = 0, u = 4, \quad x = \pm 2, u = 0, \quad du = -2x\,dx$ <br> $\int_0^2 6x\sqrt{4 - x^2}\,dx = \int_4^0 -3u^{\frac{1}{2}}\,du$ <br> $= \left[-2u^{\frac{3}{2}}\right]_4^0 = 0 - \left[-2(4)^{\frac{3}{2}}\right] = 16$ <br> $\int_{-2}^0 6x\sqrt{4 - x^2}\,dx = \left[-2u^{\frac{3}{2}}\right]_0^4 = \left[-2(4)^{\frac{3}{2}}\right] - 0 = -16$ <br> Hence $A = \lvert -16 \rvert + 16 = 32\text{ u}^2$. |
| **Specific behaviours** |
| ✓ indicates roots of graph <br> ✓ relates $du$ and $dx$ <br> ✓ integral in terms of $u$ <br> ✓ evaluates integral <br> ✓ repeats or uses symmetry to obtain second area <br> ✓ total area |

(b) The graph of $y = f(x)$ is shown at right.




Determine the value of $\int_{-1}^{1} f(1 - x^2)\,dx$.

(3 marks)

| Solution |
|---|
| Let $g(x) = 1 - x^2, -1 \le x \le 1$, so that $R_g: 0 \le y \le 1$. <br> But $R_g = D_f: 0 \le x \le 1$ and so $f(x) = 2 - 2x$ (from graph). <br> Hence $f(1 - x^2) = 2 - 2(1 - x^2) = 2x^2$ and so <br> $\int_{-1}^{1} f(x^2 - 1)\,dx = \int_{-1}^{1} 2x^2\,dx = \left[\frac{2x^3}{3}\right]_{-1}^{1} = \frac{2}{3} - \left(-\frac{2}{3}\right) = \frac{4}{3}$ |
| **Specific behaviours** |
| ✓ justifies expression for $f(x)$ <br> ✓ writes required integral <br> ✓ correct value |

**End of questions**

Supplementary page

Question number: _________